

On both parts of this table, any numbers that are similarly placed, are in proportion.

As B, F : D, O ; A, H : F, V; & which keep the same distance on the same line, are in continued proportion, as B, H, T ; D, O, Q.

This is all shown in Prop. 19, Book 5, Euclid, & by this axiom in general: If numbers in continued proportion are added or taken away from proportionals, the sums or differences are also in proportion.

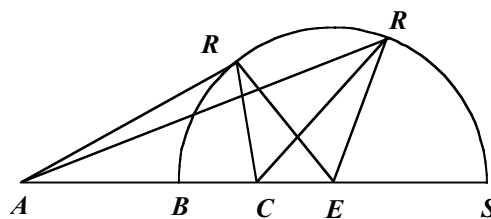
A Consequence.

Because of this, it is possible, with two given differences of third order proportionals, to find the proportionals themselves. For let H O K be three unknown proportionals, and let E, 6; F, 4 be the
58.] given differences of the same [*i.e.* E = 6, etc]: taking C, 2 the given difference, they become C, F, K or C, E, H, in continued proportion, by Prop.11, Book 6, Euclid. They are the K or H noted in the table, & K + F, or H - E are the mean O. [The reader will note how to navigate the top half of the table by adding differences, and by subtracting the same in the bottom half.]

Second Lemma³.

If three lines EC, EB, EA are in continued proportion, & with centre E and radius EB the semi-circle BRS is described, & from any point R on the periphery the straight lines RC, RA are drawn: [then] AB, BC : AR, RC are in proportion. For EA, ER, EC are proportionals from the Proposition and Definition 15, Book 1, Euclid. Therefore, the triangles AER, REC have equal angles by Prop.

6, Book 6, Euclid, or the same by Prop. 4, Book 6, Euclid. And the sides are in proportion: AE, AR : ER, RC, & alternatively, AE, ER : AR, RC. So they are [*i.e.* AR, RC], by the preceding



[Figure 21-1]

Lemma, as AE to ER or EB, thus for the difference of AB to BC; AB, BC : AR, RC are therefore proportionals. See Eutocius, Apollonius' problems in the commentary preface of the First Book:

Apollonius of Perga, On Conic Sections.

RC. Likewise the lines SA, SB, SC are proportional to the numbers 2, 3, 4. For RA, RC are proportional to AE, EC by the Second Lemma, that is, constructed from the numbers in the ratio 2:4. By the same reason, RA, RB are proportional to the lines AD, DB. That is, with the numbers in the ratio 2:3.

$10^{4/5}$	$3^{3/5}$	$1^{1/5}$	$2^{2/5}$	$4^{4/5}$	$10^{2/3}$	$5^{1/3}$	$2^{2/3}$	$5^{1/3}$	$2^{2/3}$	$22^{6/7}$	$5^{5/7}$	$1^{1/7}$	$4^{2/7}$	$12^{6/7}$
XB	DB	XD	AD	XA	NC	EC	NE	AE	NA	CO	GC	GO	BG	BO

[Table 21-2].

For if the line CB is cut in the point G, as BG to GC as 3 to 4, CB is continued to O, as OB $12^{6/7}$, .59.] OG $17^{1/7}$, OC $22^{6/7}$ are continued proportionals⁴. And with centre O, radius OG, the third arc is described, it also cuts the same points R and S. And these are the three circular arcs with centres X, N, O, on the same straight line.

And, if with the diameters staying fixed, these two circular arcs are turned in a circle [about the common line XNG]: generating spherical surfaces, of which by this motion, the common intersection is an arc, of which the diameter is RS: and this plane (containing the arc and RS) is perpendicular to the plane of the triangle ABC. And if from any point of this arc, three lines are drawn to the points A, B, C, these lines are proportional to the numbers 2, 3, 4, by the logical consequences to the Second Lemma. See the final problem in the appendix, at the end of François Viète's *Apollonius Galli*.

If anyone perhaps wants the theorem illustrated with numbers, beyond the geometrical demonstration : Given the lines AB, 6; AC, 8; CB, 10. and also given the ratios of the lines RA, RB, RC, 2, 3, 4, then AB is cut in the point D in the ratio of one and a half

If CAB had been obtuse angled, two perpendiculars might have been drawn from the points C, X to the sides AB, AC continued, which themselves will be proportional to AC, AX. With the segment of the line AC being found between the perpendicular and A, the distance of the point N to the perpendicular will have been noted, & the square of the line XN found⁵.

[i. e. $DB:AD :: 3 : 2$], as AD is $2^2/5$, DB $3^3/5$. Also AC is cut in the point E , in the ratio of two [i.e. $AE:EC :: 1: 2$], so AE is $2^2/3$, EC $5^1/3$. The segments are in continued proportion with the lengths XA, XD, XB : these proportionals are found by the consequences of the First Lemma $XA, 4^4/5$; $XD, 7^1/5$; $XB, 10^4/5$. In the same manner, $NA, 2^2/3$; $NE, 5^1/3$; $NC, 10^2/3$. In a nearby place the line XN is found; which, (as in this triangle the angles at A are right) is $_.30^{34}/225$, by Prop. 47, Book 1, Euclid. That is 5491002742 approximately. Afterwards the perpendiculars AQ, BM, CL, RP are drawn, and these lengths are to be found, together with the segments NP, NQ, NL, NM . All of this you can see expressed in numbers⁵.

AQ	<u>233108607</u>	Sq. RA	<u>92632933</u>	RA	<u>30435659</u>
NQ	<u>129504782</u>	Sq. RB	<u>208424099</u>	RB	<u>45653489</u>
NP	<u>61514772</u>	Sq. RC	<u>370531732</u>	RC	<u>60871318</u>
PQ	<u>67990010</u>	Sq. SA	<u>586612350</u>	SA	<u>76590623</u>
RP	<u>529773892</u>	Sq. SB	<u>1319877788</u>	SB	<u>114885934</u>
BM	<u>524494365</u>	Sq. SC	<u>1346449400</u>	SC	<u>153181246</u>
CL	<u>932434428</u>				
PM }	<u>456504354</u>				
PL }					

[Table 21-3].

And if with centre P , radius RP , the arc RHS is described: and we may imagine throwing up a right angle from the plane $XCBM$; as the point H projects perpendicularly above the point P : & you draw three lines to the point below H to the points A, B , and C : of these the squares will be (as before) proportional to the squares 4, 9, 16.

Sq. AH	<u>33962264</u>	AH	<u>58277152</u>
Sq. BH	<u>76415094</u>	BH	<u>87415728</u>
Sq. CH	<u>135849056</u>	CH	<u>116554304</u>

[Table 21-4].

[The last part of the Proposition: And given the base of the triangle, together with the area and the ratio of the legs; to find the legs].

For, if with the given base AB 10 of the triangle, together with the area $_.1575$, & with the ratio of the sides 3 to 2, the sides can be found [Fig. 21-3]. The given base AB of 10 parts, is cut in the given ratio by the point D , as AD 6, is to DB 4. Of these segments, the difference DX is 2, thus itself has the same ratio to DB 4 the small segment, as the same BD 4 to BC 8, the smallest of

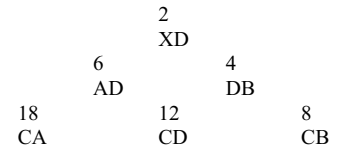
three continued proportions, by the consequences of the First Lemma. Therefore CB, 8; CD, 12; CA, 18, are continued proportions, & with centre C, radius CD, the semicircle DEF is described.

Then if the area of the given triangle $\triangle ABC$ is divided by 5, half of the given base, the quotient $\triangle EN$ is the altitude of the triangle. Which being found [proceed by] : drawing the perpendicular EN

to the base and with the same equal altitude; and through the point E is drawn EF parallel to the base AB, intersecting the semicircle drawn before in the points E and F, and the points EA, EB

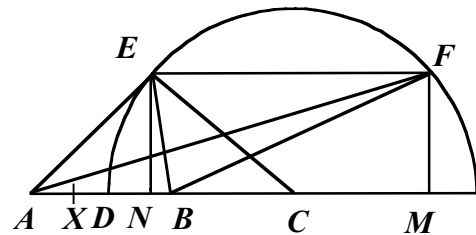
joined, likewise FA, FB. I assert that the ratios: AE to EB, & AF to FB, as AD to DB, by the Second Lemma [see Figure 21-3]: that is from this construction, as 3 to

2. And the triangles AEB, AFB are equal in area between themselves,



because of the equality of the same base. And the area of either is $\triangle EN$. The truth of all of this is shown with numbers.

<i>proportions</i>		<i>Logarithms</i>
{ XD 2 Comp. Arith.		9,69897,001
{ DB 4		0,60205,999
{ DB 4		0,60205,999
{ BC 8 -----		(1)0,90308,999



[Figure 21-3]

EN } altitude $\triangle EN$				
FM }				
Sq. EN 63				
Sq. EC 144	CM.9			
Sq. NC } 81	NC.9	Sq. EN.63	AB.10	Sq. AN.81
or MC }	CB.8	Sq. NB.1	NB.1	Sq. NE.63
	NB.1	Sq. EB.64	EB.8	AN.9
				Sq. AF.144
				AE.12
CM.9	Sq. BM.289	AC.18	Sq. AM.729	
CB.8	Sq. FM.63	CM 9	Sq. FM. 63	
BM.17	Sq. BF.352	AM.27	Sq. AF.792	

proport-ions	{ 2 3 BD.4 AD.6	proport-ions	{ { BD.4 AD.6 EB.8 EA.12 FB.352 FA.792	proport-ions	{ Sq. BD. 16 Sq. AD. 36 Sq. FB. 352 Sq. FA. 792
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[Table 21-5].

And if the altitude EN $\triangle EN$, is multiplied by $\triangle EN$, half the base AB, the product $\triangle EN$ is the area.

§21.3.

Notes on Chapter 21.

¹ The work of Apollonius has been a continued source of ideas for geometers over the centuries.

A brief account of this theorem and its extensions can be found in the *CRC Handbook of Mathematics*, p. 57, by Eric Weisstein, Chapman & Hall/CRC, 1999.

See also, perhaps, '*Excursion in Geometry*', by Stanley G. Ogilvey, Dover, pp 14 - 23. Briggs' motivation is, of course, to show how logarithms can be used to validate the geometrical reasoning for a particular triangle. As Briggs mentions, he has drawn his inspiration from Viète's work: *Appolonius Gallus*. This can be accessed in the original by the modern reader in the reprint of Schooten's collection of Viète's works: *Opera Mathematica*, p.345, problem V, (Georg Olms Verlag, 1970).

² Briggs' example amounts to the finite sequence, for $1 \leq n \leq 6$:

$a_{n+1} = \frac{3}{2}a_n$; where $a_1 = 32$, $a_2 = 48$, $a_3 = 72$, etc. In general, by induction, we have:

$a_n = a_1(p/q)^{n-1}$, where n , p , and q are positive integers, and $a_{n+1}/a_n = p/q$, the proportionality ratio.

(In the present case, $p/q = 3/2$.) In general, the differences of succeeding orders are derived sequences related to the original according to:

$$1^{\text{st}} \text{ order : } b_{n+1} = a_{n+1} - a_n = \left(\frac{p}{q} - 1\right) a_n;$$

$$2^{\text{nd}} \text{ order : } c_{n+2} = b_{n+2} - b_{n+1} = \left(\frac{p}{q} - 1\right)^2 a_n \left(= \frac{1}{4} a_n\right);$$

$$3^{\text{rd}} \text{ order : } d_{n+3} = c_{n+3} - c_{n+2} = \left(\frac{p}{q} - 1\right)^3 a_n \left(= \frac{1}{8} a_n\right);$$

$$\text{similarly } 4^{\text{th}} \text{ order : } e_{n+4} = \left(\frac{p}{q} - 1\right)^4 a_n \left(= \frac{1}{16} a_n\right),$$

$$\text{and } 5^{\text{th}} \text{ order : } f_{n+5} = \left(\frac{p}{q} - 1\right)^5 a_n \left(= \frac{1}{32} a_n = 1, \text{ when } n = 1\right).$$

We may write terms in the equivalent form: $b_{n+1} = (1 - q/p)a_{n+1}$, etc. The diagonal relations follow from these two forms, relating differing orders. These difference sequences of course, have the

same proportionality between their terms as the original sequence: the sums of succeeding orders follow a similar pattern.

³ Algebraically, if $a : b = c : d$, then $(a - c):(b - d) = a : b$, where $c < a$ and $d < b$.

See Heath, Vol. 2, *Euclid, The Thirteen Books of The Elements*, Dover (1954), p. 174, for a discussion of geometric proportionality. We note that the second lemma is related to finding the inverse point of a given point with respect to a circle with a given radius. (Note that $\triangle ECR$ is similar to $\triangle ERB$, as $EC/ER = ER/EA$, and the angle at E is common). A related discussion is to be found on pages 198 - 200, *ibid*.

The two positions of R shown by Briggs illustrate that the result is true for acute or obtuse angles at this stage: in the next section two extended sides of the triangle ABC and two circular arcs localise the points R and S.

It may be of interest to some readers to recall the extreme usefulness of this lemma when applied to physics, as a means of calculating in a simple manner from pure geometry, the gravitational force acting on a small mass outside a uniform shell of matter, as introduced by William Thompson (see, e.g. *Treatise on Natural Philosophy, Part II*, Kelvin & Tait. Cambridge, (1890), page 13 - 14) in his method of images. This is in sharp contrast to the original method used by Newton in the *Principia*, Prop. LXXI. The three dimensional aspect of the problem is thus highly suggestive of things to come: this writer has often wondered why Newton did not go down this road in the development of his gravitational theory, rather than using the horrendous method he adopted. As far as one can tell, Newton seems to have ignored Briggs' work, to his own detriment, and also robbing Briggs of a more prominent place in the evolution of mathematics at this time.

⁴ Recall that $BC = 10$, and note that neither O, G, nor the arc are shown on Figure 21-2, presumably to avoid confusion. To find these lengths: let $BO = x$, then we solve

$(x+30/7)^2 = x(x+10)$, according to Lemma 2, to give $x = BO = 12^6/7$, etc.

⁵ The extra perpendicular lines enable the lengths (RA, RB, RC), and (SA, SB, SC) to be evaluated from Pythagoras' Theorem. E.g. $RA^2 = PQ^2 + (RP - AQ)^2$. To find these, use similar triangles: e.g. $AQ = AN \cdot AX / XN$.

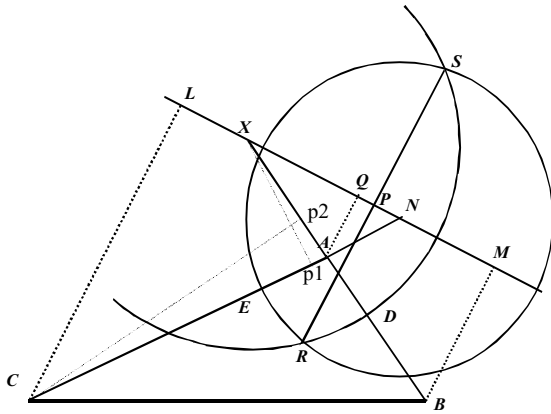


Figure 21-3.

⁶ In the case where the angle CAB is obtuse, further construction lines are necessary. Thus, if the foot of the perpendicular from X to AC is p1, and similarly C to AB p2, then the distance AP1 can be found from trigonometry, as the angle BAC is known, and similarly Xp1: hence Xp1N is a known right angled triangle, and hence XN can be found, etc.

[p.59.]

descripta tertia peripheria, transibit teiam per puncta eadem RS. Suntque harum trium peripheririum centra X, N, O, in eadem recta.

Et, si manentibus diametris, harum semiperipheriarum duae circumagantur : superficierum sphaericarum eo motu descriptarum communis intersectio erit peripheria, cuius diameter erit recta RS: eiusque planum erit perpendiculare plano trianguli ABC. Et si ab huius peripheriae puncto quolibet, ducantur tres rectae ad puncta A B C, erunt hae rectae proportionales numeris 2.3.4, per secundi Lemmatis consecarium. Vide ultimum problema, in appendicula Francisci Vietae ad finem Apollonij Galli.

Si quis forte praeter demonstrationem Geometricam, harum rerum illustrationem per numeros desideret. dantur AB, 6; AC, 8; CB, 10. dantur etiam rationes rectorum RA, RB, RC, 2.3.4 deinde secatur AB in puncto D in ratio sesquialtera, ut AD sit $2^2/5$, DB $3^3/5$. secatur etiam AC in puncto E, in ratione dupla, ut AE sit $2^2/3$, EC $5^1/3$. quae segmenta cum sint differentiae trium rectorum XA, XD, XB continue proportionalium : ipsae proportionales per consecarium primi Lemmatis inveniuntur XA, $4^4/5$; XD, $7^1/5$; XB, $10^4/5$. Eodem modo, NA, $2^2/3$; NE, $5^1/3$; NC, $10^2/3$. proximo in loco quaerenda est recta XN; quae, (cum in hoc triangulo, anguli ad A sint recti) erit $.30^{34}/225$, per 47.p.1.lib.Eucl. id est 5491002742 proxime. postea ducenda sunt perpendiculares AQ, BM, CL, RP & earum longitudines investigandae, una cum segmentis NP, NQ, NL, NM. Quae omnia hic in numeris expressa vides.

Si CAB fuisset obliquus, duae perpendiculares ducende fuissent a punctis C, X in latero AB, AC continuata, quae erunt proportionales ipsis AC, AX. & inventa segmento rectae AC inter perpendicularem & A, nota erit distantia puncti N a perpendiculari, & inveniatur quadratum rectae XN.

AQ	<u>233108607</u>	Qu. RA	<u>92632933</u>	RA	<u>30435659</u>
NQ	<u>129504782</u>	Qu. RB	<u>208424099</u>	RB	<u>45653489</u>
NP	<u>61514772</u>	Qu. RC	<u>370531732</u>	RC	<u>60871318</u>
PQ	<u>67990010</u>	Qu. SA	<u>586612350</u>	SA	<u>76590623</u>
RP	<u>529773892</u>	Qu. SB	<u>1319877788</u>	SB	<u>114885934</u>
BM	<u>524494365</u>	Qu. SC	<u>1346449400</u>	SC	<u>153181246</u>
CL	<u>932434428</u>				
PM } PL }	<u>456504354</u>				

Et si centro P, radio RP, describatur peripheria RHS: & cogitemus eius planum ad rectos angulos erectum subiecto plano XCBM; ut punctum H perpendiculariter immineat puncto P: & rectas tres ductas a puncto sublimi Had puncta A B C: earum quadrata erunt (ut priora) proportionalia quadratis 4.9.16.

Qu. AH	<u>33962264</u>	AH	<u>58277152</u>
Qu. BH	<u>76415094</u>	BH	<u>87415728</u>
Qu. CH	<u>135849056</u>	CH	<u>116554304</u>

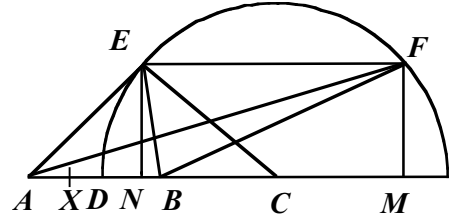
Quod, si data Basi trianguli AB 10, una cum area $.1575$, & ratione laterum 3 ad 2, quaerantur latera. Data basis AB partium 10, secetur data ratione in puncto D, ut sit AD 6, DB 4. horum segmentorum, differentia DX 2, ita se habet ad DB 4 minus segmentum, ut idem BD 4 ad BC 8 minimum e tribus continue proportionalibus: per Consecarium primi Lemmatis. Erunt igitur CB, 8; CD, 12; CA, 18, continue proportionales, & centro C, radio CD, describenda est peripheria DEF. Deinde si area trianguli data $.1575$ dividatur per 5, semissem datae basis, quotus $.63$; erit altitudo trianguli: qua inventa, ducatur EN perpendicularis basi & eidem altitudini aequalis; & per punctum E ducatur EF, parallela basi AB, intersecans peripheriam prius descriptam in punctis E F,

[p.60]

& connectantur puncta EA, EB, item FA, FB. aio AE esse ad EB, & AF ad FB, ut AD ad DB, per Secundum Lemma: id est ex fabrica, ut 3 ad 2. eruntque triangula AEB, AFB aequalia inter se, quia aequalia in eadem basi. estque area alterutrius .1575. Horum omnium veritas erit in numeris manifesta.

		2		
		XD		
	6	AD	4	
18		12		8
CA		CD		CB

<i>proportiones</i>		<i>Logarithmi</i>	
XD	2	Comp. Arith.	9,69897,001
DB	4		0,60205,999
DB	4		0,60205,999
BC	8	-----	(1)0,90308,999



EN	}	altitudo	.	63					
FM	}								
Qu. EN				63					
Qu. EC				144		CM.9			
Qu. NC				81		NC.9	Qu. EN.63	AB.10	
vel MC						CB.8	Qu. NB.1	NB.1	
						NB.1	Qu. EB.64	EB.8	
							AN.9		
								Qu. AN.81	
								Qu. NE.63	
								Qu. AF.144	
								AE.12	
<hr/>									
CM.9		Qu. BM.289		AC.18		Qu. AM.729			
CB.8		Qu. FM.63		CM.9		Qu. FM.63			
BM.17		Qu. BF.352		AM.27		Qu. AF.792			

proport-	-	ions	{	2		proport-	-	ions	{	BD.4		proport-	-	ions	{	Sq. BD. 16
				3						AD.6						Sq. AD. 36
				BD.4						EB.8						Sq. FB. 352
				AD.6						EA.12						Sq. FA. 792
										FB. .352						
										FA. .792						

Et si EN altitudo .63, multiplicet .25, semissem basis AB, factus .1575 erit area.